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Topology and its Applications 82 (1998) 227–238

TOPOLOGY AND ITS APPLICATIONS

Characterizations of submetacompactness

Heikki J.K. Junnila [a,1], Yukinobu Yajima [b,*]

[a] *Department of Mathematics, University of Helsinki, Yliopistonkatu 5, Helsinki 10, Finland*
[b] *Department of Mathematics, Kanagawa University, Yokohama 221, Japan*

Received 19 August 1996

Abstract

We give an internal characterization of submetacompactness and then we use it to prove that a space X is submetacompact if and only if $X \times (\kappa + 1)$ is suborthocompact, where κ is a cardinal no less than the Lindelöf degree of X. Similarly, we also obtain that a space X is metacompact if and only if $X \times (\kappa + 1)$ is orthocompact. © 1998 Elsevier Science B.V.

Keywords: Submetacompact; Product; Suborthocompact; ι-sequence; Rectangular open refinement; Discretely θ-expandable; Metacompact; Orthocompact

AMS classification: 54B10; 54D20

Introduction

Worrell and Wicke [14] introduced the concept of submetacompactness, which is a common generalization of metacompactness and subparacompactness. This covering property has played quite an important role in various areas, for example, in the study of generalized metric spaces.

The first author gave several characterizations of submetacompactness in [5]. In Section 1, we give another characterization under the assumption of "discrete θ-expandability".

Tamano [12] proved that a Tychonoff space X is paracompact if and only if $X \times \beta X$ is normal. Kunen obtained an analogous result by replacing βX with $\kappa + 1$, where κ is a cardinal number not less than the Lindelöf degree of X (see [8, Corollary 3.7]). The authors [4,15] have proved analogues of Tamano's theorem for metacompactness

* Corresponding author. E-mail: yuki@cc.kanagawa-u.ac.jp.
[1] E-mail: hjkjunnila@cc.helsinki.fi.

PII S0166-8641(97)00047-3

and submetacompactness. In Section 2, we prove an analogue of Kunen's result for submetacompactness.

In Section 3, we obtain a similar results for metacompactness, and we solve a problem raised in [16]. In Section 4, some additional results to [16] are stated.

All spaces are assumed to be T_1-spaces. Let κ be an infinite cardinal. For a space X, we denote by $L(X)$ the Lindelöf degree of X.

1. An internal characterization of submetacompactness

Let $\omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$; here $\omega^0 = \{()\} = \{\emptyset\}$; for all $n \in \omega$ and $s = (n_0, \ldots, n_k) \in \omega^{<\omega}$, let $s \oplus n = (n_0, \ldots, n_k, n)$.

Let X be a space and $\mathcal{V}$ an open cover of X. For $A \subset X$, let

$$\mathrm{St}(A, \mathcal{V}) = \bigcup\{V \in \mathcal{V}\colon\ V \cap A \neq \emptyset\}.$$

For $x \in X$, $\mathrm{St}(\{x\}, \mathcal{V})$ is denoted by $\mathrm{St}(x, \mathcal{V})$.

A sequence $\langle \mathcal{V}_n \rangle_{n\in\omega}$ of open covers of a space X is called a *θ-sequence* if for each $x \in X$ there is $n \in \omega$ such that $\mathcal{V}_n$ is point-finite at x. A space X is *submetacompact* if every open cover of X has a θ-sequence of open refinements.

The first author [5] proved the following:

Theorem 1.1. *A space X is submetacompact if and only if every open cover $\mathcal{U}$ of X has a sequence $\langle \mathcal{V}_n \rangle_{n\in\omega}$ of open refinements such that for each $x \in X$ there is an $n \in \omega$ and some finite set $\{U_0, \ldots, U_m\}$ of members of $\mathcal{U}$ with $\mathrm{St}(x, \mathcal{V}_n) \subset \bigcup_{i\leqslant m} U_i$ and $x \in \bigcap_{i\leqslant m} U_i$.*

A space X is *(discretely) θ-expandable* [7] if, for every locally finite (discrete) collection $\{F_\alpha\colon\ \alpha \in \kappa\}$ of closed sets in X, there is a sequence $\langle\{U_{\alpha,n}\colon\ \alpha \in \kappa\}\rangle_{n\in\omega}$ of collections of open sets in X, satisfying

(i) $F_\alpha \subset U_{\alpha,n}$ for each $\alpha \in \kappa$ and $n \in \omega$,

(ii) for each $x \in X$, there is $n \in \omega$ such that $\{U_{\alpha,n}\colon\ \alpha \in \kappa\}$ is point-finite at x.

Observe that submetacompact spaces are θ-expandable. Under the assumption of discrete θ-expandability, the condition of Theorem 1.1 can be weakened as follows:

Theorem 1.2. *A space X is submetacompact if and only if X is discretely θ-expandable and every open cover $\{U_\alpha\colon\ \alpha \in \kappa\}$ of X has a sequence $\langle \mathcal{V}_n \rangle_{n\in\omega}$ of open refinements such that for each $x \in X$ there are $\alpha \in \kappa$ and $n \in \omega$ with $x \in U_\alpha$ and $\mathrm{St}(x, \mathcal{V}_n) \subset \bigcup_{\beta\leqslant\alpha} U_\beta$.*

Proof. Since the "only if" part is obvious, we show the "if" part. For every open cover $\mathcal{U}$ of X and for each $U \in \mathcal{U}$, let

$$M(U, \mathcal{U}) = X \setminus \bigcup\{U' \in \mathcal{U}\colon\ U' \neq U\}.$$

Note that $\{M(U,\mathcal{U})\colon U \in \mathcal{U}\}$ is a discrete collection of closed sets in X, and that $\mathrm{St}(M(U,\mathcal{U}),\mathcal{U}) = U$ for each $U \in \mathcal{U}$ with $M(U,\mathcal{U}) \neq \emptyset$. It is easy to see that the two conditions in our assumption can be combined into one as follows:

($*$) *Every open cover* $\mathcal{U} = \{U_\alpha\colon \alpha \in \kappa\}$ *of* X *has a sequence* $\langle \mathcal{V}_n \rangle_{n\in\omega}$ *of open refinements such that, for each* $x \in X$, *there is* $\alpha \in \kappa$ *and* $n \in \omega$ *such that* $x \in U_\alpha$, $\mathrm{St}(x, \mathcal{V}_n) \subset \bigcup_{\beta \leqslant \alpha} U_\beta$ *and the collection* $\{U \in \mathcal{U}\colon x \in \mathrm{St}(M(U,\mathcal{U}), \mathcal{V}_n)\}$ *is finite.*

In the above situation we say, for the sake of convenience, that $\langle \mathcal{V}_n \rangle_{n\in\omega}$ is an *sm-refining sequence* for $\mathcal{U}$.

To show that X is submetacompact, let $\mathcal{U}$ be an open cover of X with cardinality κ. Write $\mathcal{U}_\emptyset = \mathcal{U} = \{U_{\alpha,\emptyset}\colon 0 < \alpha < \kappa\}$ and let $\mathcal{W}_\emptyset = \{\emptyset\}$. By induction, define open refinements $\mathcal{U}_s \cup \mathcal{W}_s$ and $\mathcal{V}_s$ of $\mathcal{U}$, where $\mathcal{U}_s = \{U_{\alpha,s}\colon 0 < \alpha < \kappa\}$, for each $s \in \omega^{<\omega} \setminus \{\emptyset\}$ as follows:

If $t \in \omega^{<\omega}$ and the open refinement $\mathcal{U}_t \cup \mathcal{W}_t$ of $\mathcal{U}$, where $\mathcal{U}_t = \{U_{\alpha,t}\colon 0 < \alpha < \kappa\}$, has been defined, then we set $U_{0,t} = \bigcup \mathcal{W}_t$, and we let $\langle \mathcal{V}_{t\oplus n} \rangle_{n\in\omega}$ be an sm-refining sequence for the open cover $\{U_{\alpha,t}\colon \alpha < \kappa\}$. For each $n \in \omega$, we let

$$U_{\alpha,t\oplus n} = U_{\alpha,t} \cap \bigcup_{\beta>\alpha} U_{\beta,t} \cap \mathrm{St}\Bigl(X \setminus \bigcup_{0\leqslant\beta<\alpha} U_{\beta,t},\ \mathcal{V}_{t\oplus n}\Bigr) \quad (\text{for } 0 < \alpha < \kappa),$$

$$\mathcal{U}_{t\oplus n} = \{U_{\alpha,t\oplus n}\colon 0 < \alpha < \kappa\} \quad \text{and}$$

$$\mathcal{W}_{t\oplus n} = \mathcal{W}_t \cup \bigl\{ \mathrm{St}\bigl(M(U,\mathcal{U}_t \cup \mathcal{W}_t), \mathcal{V}_{t\oplus n}\bigr)\colon U \in \mathcal{U}_t \bigr\}.$$

Note that the union of the collections $\mathcal{U}_{t\oplus n}$ and $\mathcal{W}_{t\oplus n}$ covers X. So $\mathcal{U}_{t\oplus n} \cup \mathcal{W}_{t\oplus n}$ is an open refinement of $\mathcal{U}$. This completes the inductive definition.

Now $\{\mathcal{U}_s \cup \mathcal{W}_s\colon s \in \omega^{<\omega}\}$ is a countable collection of open refinements of $\mathcal{U}$. To complete the proof, let $x \in X$. We show that there is $s \in \omega^{<\omega}$ such that $\mathcal{U}_s \cup \mathcal{W}_s$ is point-finite at x. Define $s_k \in \omega^{<\omega}$ and $\alpha_k < \kappa$ for each $k \in \omega$ as follows. Set $s_0 = \emptyset$. If s_k has been defined, then by definition of sm-refining sequence, there is $n \in \omega$ and $0 \leqslant \alpha_k < \kappa$ such that $x \in U_{\alpha_k,s_k}$, $\mathrm{St}(x, \mathcal{V}_{s_k\oplus n}) \subset \bigcup_{\beta\leqslant\alpha_k} U_{\beta,s_k}$ and the collection

$$\bigl\{U \in \mathcal{U}_{s_k}\colon x \in \mathrm{St}\bigl(M(U,\mathcal{U}_{s_k} \cup \mathcal{W}_{s_k}), \mathcal{V}_{s_k\oplus n}\bigr)\bigr\}$$

is finite; we now let $s_{k+1} = s_k \oplus n$. Note that it follows from the definition of the indices s_k that, for each $k \in \omega$, the collection $\mathcal{W}_{s_k}$ is point-finite at x; hence the proof is completed once we find a $k \in \omega$ such that $x \notin \bigcup \mathcal{U}_{s_k}$. Let us first note that, for each $k \in \omega$ and for every $\beta > \alpha_k$, we have that $\mathrm{St}(x, \mathcal{V}_{s_{k+1}}) \subset \bigcup_{0\leqslant\gamma<\beta} U_{\gamma,s_k}$ and hence that $x \notin U_{\beta,s_{k+1}}$. In other words, we have that $x \notin \bigcup_{\beta>\alpha_k} U_{\beta,s_{k+1}}$; as a consequence, we have that $\alpha_{k+1} \leqslant \alpha_k$. Let $i \in \omega$ be such that $\alpha_i \leqslant \alpha_j$ for each $j \in \omega$. We show that $\alpha_i = 0$. Assume on the contrary that $\alpha_i > 0$. Note that it then follows, since $x \notin \bigcup_{\beta>\alpha_i} U_{\beta,s_{i+1}}$, that $x \notin U_{\alpha_i,s_{i+2}}$. It further follows, since $\alpha_{i+2} \leqslant \alpha_i$ and $x \in U_{\alpha_{i+2},s_{i+2}}$, that we must have that $\alpha_{i+2} < \alpha_i$, and this contradicts our choice of i. Therefore, $\alpha_i = 0$. It follows that $x \notin \bigcup_{\beta>0} U_{\beta,s_{i+1}}$, in other words, that $x \notin \bigcup \mathcal{U}_{s_{i+1}}$. $\square$

In the next section, we will use Theorem 1.2 to prove a product characterization of submetacompactness. For some other applications, the following reformulation of

Theorem 1.2 in terms of increasing closed families in place of open refinements may be more natural and useful; recall that a family of sets $\{F_\alpha\colon \alpha \in \lambda\}$, indexed by an ordinal λ, is *increasing* provided that $F_\alpha \subset F_\beta$ whenever $\alpha < \beta < \lambda$.

Theorem 1.2′. *A space X is submetacompact if and only if X is discretely θ-expandable and for every open cover $\{U_\alpha\colon \alpha \in \kappa\}$ of X there exists a sequence $\langle\{F_{\alpha,n}\colon \alpha \in \kappa\}\rangle_{n\in\omega}$ of increasing closed families such that $F_{\alpha,n} \subset \bigcup_{\beta\leqslant\alpha} U_\beta$ for all $\alpha \in \kappa$ and $n \in \omega$ and, for each $x \in X$, there are $\alpha \in \kappa$ and $n \in \omega$ with $x \in F_{\alpha,n} \cap U_\alpha$.*

2. An external characterization of submetacompactness

Let X be a space and $\mathcal{V}$ a cover of X. Then let

$$\bigcap \mathcal{V}(x) = \bigcap \{V \in \mathcal{V}\colon x \in V\}$$

for $x \in X$. Let X and Y be spaces and $\mathcal{G}$ a cover of $X \times Y$. Then $\bigcap \mathcal{G}(\langle x, y\rangle)$ is denoted by $\bigcap \mathcal{G}(x, y)$ for $\langle x, y\rangle \in X \times Y$.

A cover $\mathcal{V}$ of a product space $X \times Y$ is *rectangular* if each member of $\mathcal{V}$ is of the form $U \times V$ in $X \times Y$.

A sequence $\langle \mathcal{V}_n\rangle_{n\in\omega}$ of open covers of a space X is called an *ι-sequence* [16] if for each $x \in X$ there is $n \in \omega$ such that $\bigcap \mathcal{V}_n(x)$ is a neighborhood of x in X. A space X is *suborthocompact* [15] if every open cover of X has an ι-sequence of open refinements.

Making use of Theorem 1.1, the second author [15,16] proved the following:

Theorem 2.1. *For a Tychonoff space X, the following are equivalent:*

(a) *X is submetacompact.*

(b) *$X \times \beta X$ is suborthocompact.*

(c) *Every binary open cover of $X \times \beta X$ has an ι-sequence of rectangular open refinements.*

Here, making use of our Theorem 1.2, we prove the following:

Theorem 2.2. *The following are equivalent for a space X and for $\kappa \geqslant L(X)$:*

(a) *X is submetacompact.*

(b) *$X \times (\kappa + 1)$ is suborthocompact.*

(c) *Every binary open cover of $X \times (\kappa + 1)$ has an ι-sequence of rectangular open refinements.*

Since $\kappa + 1$ is considered as a closed subset in 2^κ, Theorem 2.2 is an affirmative answer to [16, Problem 7.6]. Theorem 2.2 immediately follows from Lemmas 2.4, 2.5 and Theorem 1.2, because (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) are obvious.

Let us begin with a slight generalization of [10, Lemma 1.3]. This is necessary not for Theorem 2.2 but for Lemma 2.4(1); the proof is essentially the same as used by Scott, and we leave the details to the reader.

Proposition 2.3. *Let X be a space and C an infinite, countably compact Hausdorff space. If $X \times C$ is countably suborthocompact (i.e., every countable open cover of X has an ι-sequence of open refinements), then X is countably metacompact.*

A well-ordered sequence $\{y_\alpha\colon \alpha \in \kappa\}$ of length κ in a space Y is *right separated* if $y_\alpha \notin \mathrm{Cl}\{y_\delta\colon \delta > \alpha\}$ for each $\alpha \in \kappa$.

In particular, $\kappa + 1$ has a right separated sequence of length κ.

Lemma 2.4. *Let X be a space and C a compact Hausdorff space with a right separated sequence of length $L(X)$.*

(1) *If $X \times C$ is suborthocompact, then X is θ-expandable.*

(2) *If every binary open cover of $X \times C$ has an ι-sequence of rectangular open refinements, then X is discretely θ-expandable.*

Proof. Let $\{y_\alpha\colon \alpha \in \kappa\}$ be a right separated sequence of length κ in C, where $\kappa = L(X)$. For each $\alpha \in \kappa$, take an open neighborhood V_α of y_α in C such that

$$V_\alpha \cap \mathrm{Cl}\{y_\delta\colon \delta > \alpha\} = \emptyset.$$

Since a space is θ-expandable iff it is discretely θ-expandable and countably metacompact (see [7]), it suffices to show from Proposition 2.3 that X is discretely θ-expandable. Let $\{F_\alpha\colon \alpha \in \kappa\}$ be a discrete collection of closed sets in X. For each $\alpha \in \kappa$, let

$$G_{\alpha,0} = \Bigl(X \setminus \bigcup\{F_\beta\colon \beta \in \kappa \setminus \{\alpha\}\}\Bigr) \times V_\alpha,$$

$$G_{\alpha,1} = \Bigl(X \setminus \bigcup\{F_\beta\colon \beta \in \kappa \setminus \{\alpha\}\}\Bigr) \times \bigl(C \setminus \{y_\alpha\}\bigr).$$

(1) Let $\mathcal{G} = \{G_{\alpha,i}\colon \alpha \in \kappa \text{ and } i \in 2\}$. Since $\mathcal{G}$ is an open cover of $X \times C$, there is an ι-sequence $\langle \mathcal{H}_n \rangle_{n\in\omega}$ of open refinements of $\mathcal{G}$. For each $\alpha \in \kappa$ and $n \in \omega$, let $O_{\alpha,n} = \mathrm{St}(F_\alpha \times (C \setminus V_\alpha), \mathcal{H}_n)$ and let

$$U_{\alpha,n} = \bigl\{x \in X\colon \{x\} \times (C \setminus V_\alpha) \subset O_{\alpha,n}\bigr\} \cap U_{\alpha,n-1}.$$

Note that $U_{\alpha,n}$ is an open set in X such that $F_\alpha \subset U_{\alpha,n} \subset U_{\alpha,n-1}$ for each $\alpha \in \kappa$ and $n \in \omega$.

Now, assume that there is some $p \in X$ such that $\{U_{\alpha,n}\colon \alpha \in \kappa\}$ is not point-finite at p for each $n \in \omega$. Let $A_n = \{\alpha \in \kappa\colon p \in U_{\alpha,n}\}$ for each $n \in \omega$. Then $\langle A_n \rangle_{n\in\omega}$ is a decreasing sequence of infinite subsets in κ. So we can choose a strictly increasing sequence $\langle \alpha_n \rangle_{n\in\omega}$ in κ such that $\alpha_n \in A_n$ for each $n \in \omega$. There is a cluster point z of $\{y_{\alpha_n}\colon n \in \omega\}$ in C. Moreover, there is $m \in \omega$ such that $\bigcap \mathcal{H}_m(p,z)$ is a neighborhood of $\langle p, z\rangle$ in $X \times C$. Pick any $n \geqslant m$. By $p \in U_{\alpha_n,n} \subset U_{\alpha_n,m}$, we have

$$\langle p, z\rangle \in \{p\} \times \mathrm{Cl}\{y_{\alpha_j}\colon j > n\} \subset \{p\} \times (C \setminus V_{\alpha_n}) \subset O_{\alpha_n,m}.$$

So there is an $H_n \in \mathcal{H}_m$ which contains $\langle p, z\rangle$ and meets $F_{\alpha_n} \times (C \setminus V_{\alpha_n})$. Note that $F_{\alpha_n} \times (C \setminus V_{\alpha_n})$ does not meet $G_{\alpha,0}$ for each $\alpha \in \kappa$, and that $F_{\alpha_n} \times (C \setminus V_{\alpha_n})$ does not meet $G_{\alpha,1}$ for each $\alpha \in \kappa$ with $\alpha \neq \alpha_n$. Since $\mathcal{H}_m$ refines $\mathcal{G}$, it follows that $H_n \subset G_{\alpha_n,1}$ for each $n \geqslant m$. Since

$$\langle p, z\rangle \in \mathrm{Int}\Bigl(\bigcap \mathcal{H}_m(p, z)\Bigr) \subset \mathrm{Int}\Bigl(\bigcap_{n \geqslant m} H_n\Bigr),$$

one can choose some $k \geqslant m$ with $\langle p, y_{\alpha_k}\rangle \in \mathrm{Int}(\bigcap_{n\geqslant m} H_n)$. Hence we have

$$\langle p, y_{\alpha_k}\rangle \in H_k \subset G_{\alpha_k,1} \subset X \times \bigl(C \setminus \{y_{\alpha_k}\}\bigr).$$

This is a contradiction. Hence X is discretely θ-expandable.

(2) Let $G_i = \bigcup\{G_{\alpha,i}\colon \alpha \in \kappa\}$ for $i = 0, 1$. Since $\{G_0, G_1\}$ is a binary open cover of $X \times C$, there is an ι-sequence of $\langle \mathcal{H}_n\rangle_{n\in\omega}$ of rectangular open refinements of $\{G_0, G_1\}$. Moreover, define $\{U_{\alpha,n}\colon \alpha \in \kappa\}$ as in the proof of (1) above. Then each $U_{\alpha,n}$ is also an open set in X such that $F_\alpha \subset U_{\alpha,n} \subset U_{\alpha,n-1}$.

Now, assume that there is some $p \in X$ such that $\{U_{\alpha,n}\colon \alpha \in \kappa\}$ is not point-finite at p for each $n \in \omega$. Then, by the similar way to the above, we can choose $\{\alpha_n\colon n \in \omega\} \subset \kappa$, $z \in X$, $m \in \omega$ and $\{H_n\colon n \geqslant m\} \subset \mathcal{H}_m$. We can let $H_n = H_n' \times H_n''$ for each $n \in \omega$. Pick any $n \geqslant m$. Since $F_{\alpha_n} \times (C \setminus V_{\alpha_n})$ meets H_n and does not meet G_0, it follows that $H_n \subset G_1$. Assume $y_{\alpha_n} \in H_n''$ for some $n \geqslant m$. Since $H_n = H_n' \times H_n''$, we can pick some $x \in H_n' \cap F_{\alpha_n}$. Then we have

$$\langle x, y_{\alpha_n}\rangle \in H_n \cap \bigl(F_{\alpha_n} \times \{y_{\alpha_n}\}\bigr) \subset G_1 \cap \bigl(F_{\alpha_n} \times \{y_{\alpha_n}\}\bigr) = \emptyset.$$

This is a contradiction. Hence it follows that $y_{\alpha_n} \notin H_n''$ for each $n \geqslant m$. Similarly, we can find some $k \geqslant m$ such that $\langle p, y_{\alpha_k}\rangle \in \mathrm{Int}(\bigcap_{n\geqslant m} H_n)$. So we have $\langle p, y_{\alpha_k}\rangle \in H_k = H_k' \times H_k''$. This implies $y_{\alpha_k} \in H_k''$, which contradicts the above fact. Therefore X is discretely θ-expandable. $\square$

Lemma 2.5. *Let X be a space with $L(X) = \kappa$ such that either*

(1) *$X \times (\kappa + 1)$ is suborthocompact, or*

(2) *every binary open cover of $X \times (\kappa + 1)$ has an ι-sequence of rectangular open refinements.*

Then, for every open cover $\{U_\alpha\colon \alpha \in \kappa\}$ of X, there is a sequence $\langle\{V_{\alpha,n}\colon \alpha \in \kappa\}\rangle_{n\in\omega}$ of open covers of X such that $V_{\alpha,n} \subset U_\alpha$ for each $\alpha \in \kappa$ and $n \in \omega$ and that, for each $x \in X$, there is $n \in \omega$ and $\delta \in \kappa$ with $x \in U_\delta \setminus \bigcup_{\alpha>\delta} V_{\alpha,n}$.

Proof. Let $\{U_\alpha\colon \alpha \in \kappa\}$ be an open cover of X.

(1) Let

$$\mathcal{G} = \bigl\{U_\alpha \times (\alpha + 1), U_\alpha \times (\alpha, \kappa]\colon \alpha \in \kappa\bigr\}.$$

Then $\mathcal{G}$ is an open cover of $X \times (\kappa + 1)$. There is an ι-sequence $\langle \mathcal{H}_n\rangle_{n\in\omega}$ of open refinements of $\mathcal{G}$. For each $\alpha \in \kappa$ and $n \in \omega$, we let

$$O_{\alpha,n} = \mathrm{St}\Biggl(\Bigl(X \setminus \bigcup_{\beta<\alpha} U_\beta\Bigr) \times (\alpha, \kappa], \mathcal{H}_n\Biggr),$$

and let $V_{\alpha,n} = \{x \in U_\alpha\colon \{x\} \times (\alpha, \kappa] \subset O_{\alpha,n}\}$. Then each $\{V_{\alpha,n}\colon \alpha \in \kappa\}$ is an open cover of X such that $V_{\alpha,n} \subset U_\alpha$ for each $\alpha \in \kappa$.

Now, assume that the sequence $\langle \{V_{\alpha,n}\colon \alpha \in \kappa\}\rangle_{n\in\omega}$ does not have the desired property. Then there is some $p \in X$, satisfying

(i) $\{\alpha \in \kappa\colon p \in U_\alpha\}$ does not have a largest element,

(ii) $\{\alpha \in \kappa\colon p \in V_{\alpha,n}\}$ does not have a largest element for each $n \in \omega$,

(iii) $\sup\{\alpha \in \kappa\colon p \in U_\alpha\} = \sup\{\alpha \in \kappa\colon p \in V_{\alpha,n}\}$ for each $n \in \omega$.

Let $\lambda = \sup\{\alpha \in \kappa\colon p \in U_\alpha\}$. Let $L_n = \{\alpha \in \kappa\colon p \in V_{\alpha,n}\}$ for each $n \in \omega$. By (iii), we have $\sup L_n = \lambda$ for each $n \in \omega$. We can choose $m \in \omega$ with $\langle p, \lambda\rangle \in \operatorname{Int}(\bigcap \mathcal{H}_m(p, \lambda))$. Then take $\mu < \lambda$ with $\{p\} \times (\mu, \lambda] \subset \bigcap \mathcal{H}_m(p, \lambda)$, and pick $\delta \in L_m \cap (\mu, \lambda)$. By $p \in V_{\delta,m}$, we have $\langle p, \lambda\rangle \in \{p\} \times (\delta, \kappa] \subset O_{\delta,m}$. So there is an $H_0 \in \mathcal{H}_m$ such that $\langle p, \lambda\rangle \in H_0$ and $H_0 \cap ((X \setminus \bigcup_{\beta<\delta} U_\beta) \times (\delta, \kappa]) \neq \emptyset$. Then we have

$$\langle p, \delta\rangle \in \{p\} \times (\mu, \lambda] \subset \bigcap \mathcal{H}_m(p, \lambda) \subset H_0.$$

Hence H_0 contains the two point $\langle p, \lambda\rangle$ and $\langle p, \delta\rangle$. Since $\mathcal{H}_m$ refines $\mathcal{G}$, there is $\gamma \in \kappa$ such that either $H_0 \subset U_\gamma \times (\gamma + 1)$ or $H_0 \subset U_\gamma \times (\gamma, \kappa]$.

In case of $H_0 \subset U_\gamma \times (\gamma + 1)$, by $\langle p, \lambda\rangle \in H_0$, we have $\lambda \leqslant \gamma$. On the other hand, by $p \in U_\gamma$ and (i), we obtain $\gamma < \lambda$. This is a contradiction.

In case of $H_0 \subset U_\gamma \times (\gamma, \kappa]$, by $\langle p, \delta\rangle \in H_0$, we have $\gamma < \delta$. However, since H_0 meets $(X \setminus \bigcup_{\beta<\delta} U_\beta) \times (\delta, \kappa]$, it follows that U_γ meets $X \setminus \bigcup_{\beta<\delta} U_\beta$. Therefore, $\gamma \geqslant \delta$. This is also a contradiction.

(2) Let $G_0 = \bigcup\{U_\alpha \times (\alpha + 1)\colon \alpha \in \kappa\}$ and $G_1 = \bigcup\{U_\alpha \times (\alpha, \kappa]\colon \alpha \in \kappa\}$. Then $\{G_0, G_1\}$ is a binary open cover of $X \times (\kappa + 1)$. There is an ι-sequence $\langle \mathcal{H}_n\rangle_{n\in\omega}$ of rectangular open refinements of $\{G_0, G_1\}$. Define each $O_{\alpha,n}$ and each $V_{\alpha,n}$ as in the above proof. Then each $\{V_{\alpha,n}\colon \alpha \in \kappa\}$ is also an open cover of X such that $V_{\alpha,n} \subset U_\alpha$ for each $\alpha \in \kappa$.

Assume that the sequence $\langle \{V_{\alpha,n}\colon \alpha \in \kappa\}\rangle_{n\in\omega}$ does not have the desired property. Then there is some $p \in X$ satisfying conditions (i)–(iii) appearing in the first part of this proof. We define $\lambda \leqslant \kappa$, $L_n \subset \kappa$, $m \in \omega$, $\delta \leqslant \lambda$ and $H_0 \in \mathcal{H}_m$ similarly as in the first part of the proof. Note that $\{\langle p, \lambda\rangle, \langle p, \delta\rangle\} \subset H_0$, and that $H_0 \subset G_0$ or $H_0 \subset G_1$. We may let $H_0 = H_0' \times H_0''$. Now, we show that $H_0' \subset \bigcup_{\beta<\delta} U_\beta$. Assuming the contrary, pick $x \in H_0' \setminus \bigcup_{\beta<\delta} U_\beta$. Then note that $\langle x, \delta\rangle \notin G_1$. By $\langle p, \delta\rangle \in H_0$, we have $\langle x, \delta\rangle \in H_0' \times H_0'' = H_0$. Hence we obtain $H_0 \not\subset G_1$, which means $H_0 \subset G_0$. So we can find $\gamma \in \kappa$ with $\langle p, \lambda\rangle \in U_\gamma \times (\gamma + 1)$. This means $\lambda \leqslant \gamma$. However, by $p \in U_\gamma$ and (i), we have $\gamma < \sup\{\alpha \in \kappa\colon p \in U_\alpha\} = \lambda$. This is a contradiction. Hence we conclude that $H_0' \subset \bigcup_{\beta<\delta} U_\beta$. On the other hand, it follows, since

$$H_0 \cap \left(\left(X \setminus \bigcup_{\beta<\delta} U_\beta\right) \times (\delta, \kappa]\right) \neq \emptyset,$$

that $H_0' \not\subset \bigcup_{\beta<\delta} U_\beta$. This is also a contradiction. $\square$

3. Characterizations of metacompactness

A space X is *metacompact* if every open cover of X has a point-finite open refinement.

A space X is *almost (discretely) expandable* [11] if, for every locally finite (discrete) collection $\{F_\alpha\colon \alpha \in \kappa\}$ of closed sets in X, there is a point-finite collection $\{U_\alpha\colon \alpha \in \kappa\}$ of open sets in X such that $F_\alpha \subset U_\alpha$ for each $\alpha \in \kappa$.

Note that a space is almost expandable iff it is almost discretely expandable and countably metacompact (see [11, Theorem 2.8(ii)]).

By a result of [2] (see also [11]), a space X is metacompact provided X is submetacompact and almost discretely expandable. Hence the following characterization of metacompactness can be derived from Theorem 1.2.

Theorem 3.1. *A space X is metacompact if and only if X is almost discretely expandable and every open cover $\{U_\alpha\colon \alpha \in \kappa\}$ of X has an open refinement $\mathcal{V}$ such that for each $x \in X$ there is $\alpha \in \kappa$ with $x \in U_\alpha$ and* $\mathrm{St}(x, \mathcal{V}) \subset \bigcup_{\beta \leqslant \alpha} U_\beta$.

The above result can also be stated in the following form:

Theorem 3.1′. *A space X is metacompact if and only if X is almost discretely expandable and for every open cover $\{U_\alpha\colon \alpha \in \kappa\}$ of X there exists an increasing closed cover $\{F_\alpha\colon \alpha \in \kappa\}$) such that $F_\alpha \subset \bigcup_{\beta \leqslant \alpha} U_\beta$ for every $\alpha \in \kappa$ and, for each $x \in X$, there exists $\alpha \in \kappa$ with $x \in F_\alpha \cap U_\alpha$.*

Let us note that one can derive similar characterizations for paracompactness with the help of the result from [11] that a space is paracompact iff the space is submetacompact and "discretely expandable".

An open cover $\mathcal{V}$ of a space X is *interior-preserving* if $\bigcap \mathcal{V}'$ is open in X for each $\mathcal{V}' \subset \mathcal{V}$. A space X is *orthocompact* if every open cover of X has an interior-preserving open refinement.

Lemma 3.2. *Let X be a space and C a compact Hausdorff space with a right separated sequence of length $L(X)$.*

(1) *If $X \times C$ is orthocompact, then X is almost expandable.*

(2) *If every binary open cover of $X \times C$ has an interior-preserving rectangular open refinements, then X is almost discretely expandable.*

The proof of Lemma 3.2 is simpler than that of Lemma 2.4. So it is omitted.

Since submetacompact, almost discretely expandable spaces are metacompact, the following is an immediate consequence of Lemma 3.2.

Theorem 3.3. *Let X be a submetacompact space and C a compact Hausdorff space with a right separated sequence of length $L(X)$. Then the following are equivalent:*

(a) *X is metacompact.*

(b) *$X \times C$ is orthocompact.*

(c) *Every binary open cover of $X \times C$ has an interior-preserving rectangular open refinements.*

Note that the condition on the compact space C appearing in the above theorem can also be phrased by saying that the hereditary Lindelöf degree of C is at least as big as the Lindelöf degree of X, in symbols, $hL(C) \geqslant L(X)$.

It is natural to inquire, whether submetacompactness of X is necessary for Theorem 3.3 to hold. In particular, the following problem is open.

Problem 3.4. Is a space X metacompact provided there exists a compact Hausdorff space C such that $hL(C) \geqslant L(X)$ and $X \times C$ is orthocompact?

Note that a positive solution to Problem 3.4 would provide a strengthening of several known results dealing with orthocompact product spaces with a compact factor; see, e.g., [1,4,10]. It would also provide a strengthening of a part of the following result; it follows from Theorems 2.2 and 3.3 and gives a solution to [16, Problem 7.5]:

Theorem 3.5. *The following are equivalent for a space X and for $\kappa \geqslant L(X)$:*

(a) *X is metacompact.*

(b) *$X \times (\kappa + 1)$ is orthocompact.*

(c) *Every binary open cover of $X \times (\kappa + 1)$ has an interior-preserving rectangular open refinements.*

A space X is *κ-metacompact* (*κ-orthocompact*) if every open cover of X with cardinality $\leqslant\kappa$ has a point-finite (an interior-preserving) open refinement. Similarly, almost (discretely) κ-expandability is defined.

Recall that well-ordered sequence $\{y_\alpha\colon \alpha \in \kappa\}$ of length κ in a space Y is a *free sequence* if

$$\mathrm{Cl}\{y_\beta\colon \beta < \alpha\} \cap \mathrm{Cl}\{y_\gamma\colon \alpha \leqslant \gamma < \kappa\} = \emptyset$$

for each $\alpha \in \kappa$.

Since a free sequence is right separated and $\kappa + 1$ has a free sequence of length κ, the following is not only a partial answer to Problem 3.4 but also a slight generalization of Theorem 3.5.

Theorem 3.6. *Let X be a space and C a compact Hausdorff space with a free sequence of length κ. Then the following are equivalent:*

(a) *X is κ-metacompact.*

(b) *$X \times C$ is κ-orthocompact.*

(c) *Every binary open cover of $X \times C$ has an interior-preserving rectangular open refinements.*

Seeing the discussion above, it is not difficult to give the proof of Theorem 3.6; the details are left to the reader.

Remark. Let C be a compact Hausdorff space with a free sequence of length κ. Then there is a closed subset F in C and a continuous (perfect) map of F onto $\kappa+1$ (see [8, Lemma 3.17]). However, different from normality, orthocompactness is not preserved under perfect maps (see [3]). So, Theorem 3.6 may not be necessarily an immediate consequence of Theorem 3.5.

4. Other characterizations

Recall that a space X is *subnormal* if for any two disjoint closed subsets F_0 and F_1 of X there are disjoint G_δ-sets G_0 and G_1 such that $F_i \subset G_i$ for $i = 0, 1$.

A space X is *collectionwise δ-normal* [6] if, for every discrete collection $\{F_\alpha\colon\ \alpha \in \kappa\}$ of closed sets in X, there is a disjoint collection $\{G_\alpha\colon\ \alpha \in \kappa\}$ of G_δ-sets such that $F_\alpha \subset G_\alpha$ for each $\alpha \in \kappa$.

The following is essentially due to Rudin [9].

Lemma 4.1. *Let X be a space and C a compact Hausdorff space with weight $\geqslant L(X)$. If $X \times C$ is subnormal, then X is collectionwise δ-normal.*

Proof. Since the proof is obtained by modifying that of [9, Theorem 2], let us use the similar notations as in there. Let $\{G_\alpha\colon\ \alpha \in \lambda\}$ be a discrete collection of closed sets in X, where $\lambda = L(X)$. We only choose disjoint G_δ-sets $\bigcap_{n\in\omega} O_n$ and $\bigcap_{n\in\omega} Q_n$, where each O_n and each Q_n are open in $X \times C$, instead of the disjoint open sets O and Q in $X \times C$. Moreover, we similarly define $O_{\alpha n}$, $O'_{\alpha n}$, $Q_{\alpha n}$ and $Q'_{\alpha n}$ as O_α, O'_α, Q_α and Q'_α, respectively. Let $M_{\alpha n} = O_{\alpha n} \cap O'_{\alpha n} \cap Q_{\alpha n} \cap Q'_{\alpha n}$ for each $\alpha \in \lambda$ and $n \in \omega$. Then $\{\bigcap_{n\in\omega} M_{\alpha n}\colon\ \alpha \in \lambda\}$ is a disjoint collection of G_δ-sets in $X \times C$ such that $G_\alpha \subset \bigcap_{n\in\omega} M_{\alpha n}$ for each $\alpha \in \lambda$. $\square$

In the following γX denotes a compactification of X.

Lemma 4.2. *A Tychonoff space X is submetacompact (paracompact) if and only if for each compact subset K of $\gamma X \setminus X$, there is a σ-closure-preserving closed cover $\mathcal{F}$ of X such that $\mathrm{Cl}_{\gamma X} F \cap K = \emptyset$ for each $F \in \mathcal{F}$ (and $\{\mathrm{Int}\, F\colon\ F \in \mathcal{F}\}$ covers X).*

This follows from [5, Theorem 4.4] as well as [16, Lemma 2.3]. The parenthetical part follows from [4, Theorem 3.4]. We use this lemma instead of the definitions of submetacompactness and paracompactness.

We improve [16, Theorem 4.1] as follows:

Theorem 4.3. *A Tychonoff space X is subparacompact if and only if every binary open cover of $X \times \gamma X$ has a σ-closure-preserving rectangular closed refinement.*

Proof. We only show the "if" part. Since every binary open cover of $X \times \gamma X$ has a countable closed refinement, $X \times \gamma X$ is subnormal. It suffices to show from Lemma 4.1 and [6, Theorem 2.7] that X is submetacompact.

Let K be a compact subset of $\gamma X \setminus X$. Take the binary open cover

$$\mathcal{G} = \{X \times (\gamma X \setminus K), (X \times \gamma X) \setminus \Delta\}$$

of $X \times \gamma X$, where $\Delta = \{(x, x)\colon x \in X\}$. There is a σ-closure-preserving rectangular closed refinement $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$ of $\mathcal{G}$. Let $F = C_F \times D_F$ for each $F \in \mathcal{F}$. Let $\mathcal{C}_n = \{C_F \cap D_F\colon F \in \mathcal{F}_n\}$ for each $n \in \omega$, and let $\mathcal{C} = \bigcup_{n\in\omega} \mathcal{C}_n$. Note that $F \cap \Delta$ is homeomorphic to $C_F \cap D_F$ for each $F \in \mathcal{F}$ and that Δ is homeomorphic to X. Since $\{F \cap \Delta\colon F \in \mathcal{F}_n\}$ is closure-preserving in Δ, so is $\mathcal{C}_n$ in X for each $n \in \omega$. It is easy to see that $\mathcal{C}$ covers X. Hence $\mathcal{C}$ is a σ-closure-preserving closed cover of X. Notice that $F \in \mathcal{F}$ meets Δ if $C_F \cap D_F \neq \emptyset$. Since $\mathcal{F}$ refines $\mathcal{G}$, D_F does not meet K for each $F \in \mathcal{F}$ which meets Δ. For each $F \in \mathcal{F}$, we have

$$\mathrm{Cl}_{\gamma X}(C_F \cap D_F) \cap K \subset D_F \cap K = \emptyset.$$

Hence it follows from Lemma 4.2 that X is submetacompact. □

Recall that a space X is *countably subparacompact* if every countable open cover of X has a countable closed refinement.

The statement of [16, Theorem 4.3] is so clumsy that we rewrite it here.

Proposition 4.4. *A countably subparacompact space X is subparacompact if and only if every binary open cover of $X \times (\kappa + 1)$ has a σ-locally finite rectangular closed refinement, where $\kappa \geqslant L(X)$.*

Proof. Let $\mathcal{U}$ be a well-monotone open cover of X with cardinality $\leqslant\kappa$. Assume the case $\mathrm{cf}(\kappa) = \omega$. Since $\mathcal{U}$ is well-monotone, there is a countable increasing subcover $\mathcal{U}_0$ of $\mathcal{U}$. Since X is countably subparacompact, $\mathcal{U}_0$ has a countable closed refinement. For the case $\mathrm{cf}(\kappa) > \omega$, the same argument as in the proof of [16, Theorem 4.3] shows that $\mathcal{U}$ has a σ-locally finite closed refinement. □

Remark. It was proved in [13, Theorem 2.7] that "$X \times \gamma X$" in Theorem 4.3 can be replaced by "$X \times 2^{\kappa}$". However, we do not know whether "σ-locally finite" in Proposition 4.4 can be replaced by "σ-closure-preserving".

Using the parenthesized part of Lemma 4.2 instead of [16, Lemma 2.3], we can also get an improvement of [16, Theorem 3.1] which is an affirmative answer to [16, Problem 7.1].

Theorem 4.5. *A Tychonoff space X is paracompact if and only if every binary open cover of $X \times \gamma X$ has a σ-closure-preserving rectangular open refinement.*

Proof. We only show the "if" part. Let K and $\mathcal{G}$ be the same ones as in the proof of Theorem 4.3. There is a σ-closure-preserving rectangular open refinement $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$

of $\mathcal{G}$. We may assume that $\mathcal{V}_n \subset \mathcal{V}_{n+1}$ for each $n \in \omega$. Let $V = U_V \times W_V$ for each $V \in \mathcal{V}$. Let

$$\Lambda_n = \left\{ \lambda \subset \mathcal{V}_n \colon \lambda \text{ is finite, } \bigcap_{V \in \lambda} U_V \neq \emptyset \text{ and } \bigcup_{V \in \lambda} W_V = \gamma X \right\}$$

for each $n \in \omega$. Let $F_\lambda = \bigcap_{V \in \lambda} \operatorname{Cl} U_V$ for each $\lambda \in \Lambda_n$, $n \in \omega$. Let $\mathcal{F}_n = \{F_\lambda \colon \lambda \in \Lambda_n\}$ for each $n \in \omega$. Since each $\mathcal{V}_n$ is closure-preserving, so is $\{\operatorname{Cl} U_V \colon V \in \mathcal{V}_n\}$. Hence each $\mathcal{F}_n$ is also closure-preserving in X. Since γX is compact, it is easy to verify that $\{\operatorname{Int} F \colon F \in \mathcal{F}_n,\ n \in \omega\}$ covers X. Note that if $V \in \mathcal{V}$ is disjoint from Δ, then $\operatorname{Cl}_{\gamma X} U_V \cap W_V = \emptyset$. Hence we have

$$\operatorname{Cl}_{\gamma X} F_\lambda \cap K \subset \left(\bigcap_{V \in \lambda} \operatorname{Cl}_{\gamma X} U_V \right) \cap \left(\bigcup \{ W_V \colon V \in \lambda \text{ with } V \cap \Delta = \emptyset \} \right) = \emptyset$$

for each $\lambda \in \Lambda_n$, $n \in \omega$. Thus $\bigcup_{n \in \omega} \mathcal{F}_n$ satisfies the condition in the parenthetical part of Lemma 4.2. Hence X is paracompact. □

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